Comment on "Existence of internal modes of sine-Gordon kinks"

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In Phys. Rev. B **42**, 2290 (1990) we used a rigorous projection operator collective variable formalism for nonlinear Klein-Gordon equations to prove the continuum sine-Gordon (SG) equation has a long lived quasimode whose frequency $\omega_s = 1.004\Gamma_0$ is in the continuum just above the lower phonon band edge with a lifetime $(1/\tau_s) = 0.0017\Gamma_0$. We confirmed the analytic calculations by simulations which agreed very closely with the analytic results. In Phys. Rev. E **62**, R60 (2000) the authors performed two numerical investigation which they asserted "show that neither intrinsic internal modes nor quasimodes exist in contrast to previous results." In this paper we prove their first numerical investigation could not possibly observe the quasimode in principle and their second numerical investigation actually demonstrates the existence of the SG quasimode. Our analytic calculations and verifying simulations were performed for a stationary sine-Gordon soliton fixed at the origin. Yet the authors in Phys. Rev. E **62**, R60 (2000) state the explanation of our analytic simulations and confirming simulations are due to the Doppler shift of the phonons emitted by our stationary sine-Gordon soliton which thus has a zero Doppler shift.

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I. INTRODUCTION

In Ref. [1] we proved the continuum sine-Gordon (SG) equation has a long lived quasimode whose frequency from simulation is $\omega_s = (1.004 \pm 0.001)\Gamma_0$ (where Γ_0 is the frequency of the lower band edge in units where the speed of sound $c=1$) and whose lifetime from simulation is $(1/\tau_s) = (0.003 \pm 0.001)\Gamma_0$. We used a rigorous projection operator collective variable (CV) formalism for nonlinear Klein-Gordon equations derived in Ref. [2] to calculate the quasimode frequency and lifetime. Our calculated theoretical values for the frequency and inverse lifetime are $\Omega = 1.00585\Gamma_0$ and $(1/\tau_s) = 0.0017\Gamma_0$ which agree very well with our simulation values.

In Ref. [3] the authors performed two numerical investigations for the SG in which they assert "show that neither intrinsic internal modes nor quasimodes exist in contrast to previous results" referring to Ref. [1]. In Secs. III and IV we analyze their two numerical investigations in detail and prove their first numerical investigation could not possibly observe the quasimode in principle and that their second numerical investigations actually observes the SG quasimode at the beginning of their simulation. However, the length of their system was so short for their long observation time that there were many transversals of the system by phonons emitted at different times by the soliton, which then reflected from the end of the system and then interfered with phonons emitted later. Thus each phonon interfered with phonons emitted earlier and phonons emitted later, which led to a very complicated interference pattern. The authors of Ref. [3] concluded that the complicated interference pattern was a "proof" that SG quasimodes do not exist. However, the correct conclusion is that their poorly designed numerical investigation was for a time t that was more than ten times too long for the length of their system to avoid the irrelevant interferences. In the first 200 s just before the first emitted phonons reflected off the end of the system and returned to the stationary emitting SG soliton, the finite lifetime of the quasimode is clearly observed.

Our analytic calculations and verifying simulations were all for a continuum, force free and stationary SG soliton i.e., the center of mass of the SG was fixed at the origin for all times. However, in their two numerical investigations in Ref. [3] the authors provided an explanation of our analytic calculations and verifying simulations, which was that our phonons were Doppler shifted. This is not possible for phonons emitted by a stationary SG soliton fixed at the origin. As a result, their two numerical investigations and their "explanation" of our results have no relevance to the validity of our analytic soliton and verifying simulations of our continuum stationary and force free SG quasimode.

In Sec. II we outline the derivation of the exact equations of motion for the SG equation. We prove in Sec. III that the first numerical search for the SG quasimode in Ref. [3] could not observe the SG quasimode in principle. While in Sec. IV we show that in their second numerical investigation the authors of Ref. [3] actually observe the SG quasimode at the beginning of their simulation. However, their simulation was for a time too large for the length of their system. Consequently, they observed a complicated interference pattern which was irrelevant in the SG quasimode that was clearly observable at the beginning of their simulation. In Sec. VI we present our conclusions and discuss a recent work, Ref. [5], which contains a solution of the SG equation by using the inverse transform method and find our SG quasimode solution is valid.

II. EQUATIONS OF MOTION FOR THE SG QUASIMODE

The purpose of this section is to outline the derivation of the equations of motion from our exact CV equations for the SG equation in Ref. [1] that we actually solved for the SG quasimode in Ref. [1]. We need these rigorous equations in order to contrast them with the two CV equations of motion for the kink momentum $P(t)$ and width of the kink $l(t)$ which form the basis of their theoretical analysis of our equations of motion and which we repeat below in Eqs. (6) and (8).

Where we show their equation of motion for $P(t)$ is irrelevant to our derivations of the SG quasimode and their equation of motion for $\Gamma(t)$ is independent of the phonon dressing whose interaction with χ gives rise to the SG quasimode. The slope $\Gamma(t)=2\pi[l(t)]^{-1}$.

We start with the equation of motion for $X(t)$, $\Gamma(t)$, and $\chi(t)$ whose solutions are rigorously equivalent to the solution of the SG partial differential equation

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\left[\frac{\pi}{l_0}\right]^2\sin\phi=0, \tag{1}$$

where

$$\phi(x,t)=\sigma[\xi(t)]+\chi[\xi(t),t], \tag{2}$$

and

$$\xi(t)\equiv\Gamma(t)[x-X(t)]. \tag{3}$$

The soliton solution σ is

$$\sigma(\xi)=4\tan^{-1}\exp[\xi]. \tag{4}$$

$\chi(t)$ is the solution of

$$\begin{aligned}\frac{\partial^2\chi}{\partial t^2}-\chi''\Bigg[&\Gamma^2(1-\dot{X}^2)\\&+2\xi\dot{X}\dot{\Gamma}-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\Bigg]+2\frac{\partial\chi'}{\partial t}\left[\left[\frac{\dot{\Gamma}}{\Gamma}\right]\xi-\dot{X}\Gamma\right]\\&+\chi'\left[\left[\frac{\ddot{\Gamma}}{\Gamma}\right]\xi-2\dot{X}\dot{\Gamma}-\ddot{X}\Gamma\right]+\frac{\partial V(\sigma+\chi)}{\partial\sigma}\\=\sigma''&\left[\Gamma^2(1-\dot{X}^2)+2\xi\dot{X}\dot{\Gamma}-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\right]\\&-\sigma'\left[\left[\frac{\ddot{\Gamma}}{\Gamma}\right]\xi-2\dot{X}\dot{\Gamma}-\ddot{X}\Gamma\right].\end{aligned} \tag{5}$$

The equation of motion for $\ddot{X}(t)$ is

$$\begin{aligned}\ddot{X}=-\frac{\dot{X}\dot{\Gamma}}{\Gamma(1-b_X)}-\frac{1}{M_X(1-b_X)}\Bigg[&\langle\sigma'|\chi''\rangle\Gamma^2(1-\dot{X}^2)\\&+2\langle\sigma'|\xi\chi''\rangle\dot{X}\dot{\Gamma}-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\langle\sigma'|\xi^2\chi''\rangle\\&-2\frac{\dot{\Gamma}}{\Gamma}\left\langle\sigma'|\xi\frac{\partial\chi'}{\partial t}\right\rangle+2\dot{X}\Gamma\left\langle\sigma'|\frac{\partial\chi'}{\partial t}\right\rangle-\frac{\ddot{\Gamma}}{\Gamma}\langle\sigma'|\xi\chi'\rangle\\&+2\langle\sigma'|\chi'\rangle\dot{X}\dot{\Gamma}-\left\langle\sigma'|\frac{\partial V}{\partial\sigma}\right\rangle\Bigg],\end{aligned} \tag{6}$$

where

$$M_X\equiv\Gamma\langle\sigma'|\sigma'\rangle \tag{7}$$

is the bare mass of the kink associated with the X motion, and $b_X\equiv\Gamma\langle\sigma''|\chi\rangle/M_X$. The dot product $\langle f|g\rangle$ is defined as

$$\langle f|g\rangle\equiv\int f^*(\xi)g(\xi)d\xi.$$

The equation of motion for $\Gamma(t)$ is

$$\begin{aligned}\ddot{\Gamma}=&\frac{3\dot{\Gamma}^2}{2\Gamma(1-b_\Gamma)}-\frac{M_X(1-\dot{X}^2)}{2\Gamma M_\Gamma(1-b_\Gamma)}\\&+\frac{1}{M_\Gamma(1-b_\Gamma)}\Bigg[\langle\xi\sigma'|\chi''\rangle(1-\dot{X}^2)+\frac{2\dot{X}\dot{\Gamma}}{\Gamma^2}\langle\xi\sigma'|\chi''\xi\rangle\\&-\frac{1}{\Gamma^2}\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\langle\xi\sigma'|\xi^2\chi''\rangle-\frac{2}{\Gamma^2}\frac{\dot{\Gamma}}{\Gamma}\left\langle\xi\sigma'|\xi\frac{\partial\chi'}{\partial t}\right\rangle\\&+\frac{2\dot{X}}{\Gamma}\left\langle\xi\sigma'|\frac{\partial\chi'}{\partial t}\right\rangle+\left[\frac{2\dot{X}\dot{\Gamma}}{\Gamma^2}+\frac{\ddot{X}}{\Gamma}\right]\langle\xi\sigma'|\chi'\rangle\\&-\frac{1}{\Gamma^2}\left\langle\xi\sigma'|\frac{\partial V}{\partial\sigma}\right\rangle\Bigg],\end{aligned} \tag{8}$$

where

$$M_\Gamma\equiv\Gamma^{-3}\langle\xi\sigma'|\xi\sigma'\rangle. \tag{9}$$

Since the center of mass motion does not play any role in the existence of the SG quasimode we set $X(t)\equiv0$ in the equations of motion for $\chi(t)$ and $\Gamma(t)$. We are interested in small oscillations of the quasimode so we linearize Eqs. (5) and (8) to first order in χ and obtain

$$\begin{aligned}\frac{\partial^2\chi}{\partial t^2}-\chi''&\left[\Gamma^2-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\right]+2\frac{\dot{\Gamma}}{\Gamma}\frac{\partial\chi'}{\partial t}\xi+\frac{\ddot{\Gamma}}{\Gamma}\xi\chi'\\&+\Gamma_0^2\sin\sigma+\Gamma_0^2\chi\cos\sigma\\&=\sigma''\left[\Gamma^2-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\right]-\xi\sigma'\frac{\ddot{\Gamma}}{\Gamma},\end{aligned} \tag{10}$$

and

$$\begin{aligned}\ddot{\Gamma}=&\left[\frac{3\dot{\Gamma}^2}{2\Gamma}-\frac{M_X}{2\Gamma M_\Gamma}\right](1+b_\Gamma)+\frac{\langle\xi\sigma'|\chi''\rangle}{M_\Gamma}\\&-\frac{1}{M_\Gamma\Gamma^2}\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\langle\xi\sigma'|\xi^2\chi''\rangle-\frac{2}{M_\Gamma\Gamma^2}\frac{\dot{\Gamma}}{\Gamma}\left\langle\xi\sigma'|\xi\frac{\partial\chi'}{\partial t}\right\rangle\\&-\frac{\Gamma_0^2}{M_\Gamma\Gamma^2}\langle\xi\sigma'|\sin\sigma\rangle(1+b_\Gamma)-\frac{\Gamma_0^2}{M_\Gamma\Gamma^2}\langle\xi\sigma'|\chi\cos\sigma\rangle.\end{aligned} \tag{11}$$

Since we are considering only small oscillations in χ we further linearize Eqs. (10) and (11) in $\delta\Gamma\equiv\Gamma(t)-\Gamma_0$. Finally we obtain

$$\frac{\partial^2\chi}{\partial t^2}-\Gamma_0^2\chi''+\Gamma_0^2\chi\cos\sigma_0=2\Gamma_0\delta\Gamma\sigma_0''-\xi_0\sigma_0'\frac{\delta\ddot{\Gamma}}{\Gamma_0}, \tag{12}$$

and

$$\delta\ddot{\Gamma} = -\Omega_{\mathrm{SG}}^{2}\delta\Gamma + \frac{1}{M_{\Gamma_0}}\langle\xi_0\sigma_0'|\chi''\rangle - \frac{1}{M_{\Gamma_0}}\langle\xi_0\sigma_0'|\chi\cos\sigma_0\rangle, \tag{13}$$

where

$$\sigma_0 \equiv \sigma|_{\Gamma=\Gamma_0}, \quad \xi_0 \equiv \Gamma_0 x.$$

In the remainder of Ref. [1] we solved these equations of motion analytically and calculated the lifetime of the quasimode.

III. FIRST NUMERICAL INVESTIGATION

The first numerical search for the SG quasimode in Ref. [3] consisted of trying to find the SG quasimode by measuring numerically the absorption spectrum of a discrete SG equation driven by an ac field. What they measured was the phonon absorption spectrum of the linearized discrete SG equation in an ac field which is

$$\begin{aligned}\ddot{\chi}_n - (\chi_{n+1} + \chi_{n-1} - 2\chi_n) + \Gamma_0^2\chi_n \\ = (\sigma_{n+1} + \sigma_{n-1} - 2\sigma_n)\dot{X}^2 + f(t),\end{aligned} \tag{14}$$

where σ_n is the discrete SG soliton at position n, and χ_n is the discrete SG phonon at position n. The external ac field is $f(t)=\epsilon\exp(i\omega t)$ and $\dot{X}(t)$ is the velocity of the center of mass of the SG soliton. We point out again our derivations and simulations were for a stationary continuum SG where $\dot{X}(t)\equiv 0$. The spectrum they obtained by numerically solving the ac driver discrete SG equation is given in their Fig. 1 namely

$$\omega_n = \left[1 + \left(\frac{2\pi n}{l}\right)^2\right]^{1/2} \quad \text{for } n = 1,2,\ldots,N, \tag{15}$$

which is just the spectrum one obtains by solving Eq. (1) analytically. For $f(t)=\epsilon\cos\omega t$ and $\dot{X}(t)=0$ the spectrum is $\Sigma_n\delta(\omega-\omega_n)$. If you include $X(t)$ you get exactly the same spectrum by taking $f(t)=\cos[(\omega/2)t]$ because $P^2(t)$ is then proportional to $\cos[2(\omega/2)t]$, which also yields the identical spectrum $\Sigma_n\delta(\omega-\omega_n)$. This is what they observe and what one obtains by analytically solving Eq. (14). Equation (14), which is the basis of their first numerical investigation, is never mentioned in Ref. [3]; only the numerically observed spectrum Eq. (15) is presented.

What is most important about their first numerical investigation, is that it could not possibly detect the quasimode even in principle. In order to observe a quasimode in absorption it must first be created and then observed during its finite lifetime. Consequently, a quasimode is usually observed in emission. The SG quasimode can be excited as an initial condition by deforming the slope or the width of the kink as an initial condition as we did in our derivations and simulations in Ref. [1] and as the authors of Ref. [3] did in their second numerical investigation which we discuss in the next section. The quasimode can also be excited by any potential that distorts the slope of width of the SG soliton. The force on the slope $\Gamma(t)$ due to a potential $V(x)$ is

$$F = \int dx\, V_{,\sigma}(x)\frac{\partial\sigma}{\partial\Gamma}. \tag{16}$$

For an ac field $V_{,\sigma}=f(t)$, so the force F vanishes because $f(t)\int dx(\partial\sigma/\partial\Gamma)=0$. Thus an ac field cannot possibly excite a phonon mode. Consequently, their first numerical investigation in Ref. [3] could not possibly detect the presence of the SG quasimode and thus it has no relevance to the existence or nonexistence of the SG quasimode. It is important to stress that a quasimode is different from an eigenmode of a linearized Klein-Gordon equation in that an unoccupied eigenmode exists even if it is unoccupied, whereas a quasimode has first to be created in order to be observed and it lasts only for its lifetime.

IV. SECOND NUMERICAL INVESTIGATION

In Ref. [1] we performed simulations that verified our analytic solutions for the SG quasimode. We performed three simulations for $\Gamma(t)$ and $\chi(t)$ for a stationary SG soliton. We also simulated the Fourier transform of $\Gamma(t)$ which gives the quasimode frequency and lifetime. The simulations agree very closely with analytic results. We considered cases where the initial slope was different from Γ_0, i.e., $\delta\Gamma(0)\neq 0$ and for $\delta\dot{\Gamma}(0)\neq 0$. We considered two cases where the length of the system was 1000 units and a third case where the length of the system was 200 units. For the system of length 1000 we followed the time development of $\Gamma(t)$ and $\chi(t)$ for times t that were short compared with the time a spontaneously emitted phonon would travel to the end of the system reflect and interfere with phonons emitted later. In the third case we took a short system $L=200$ and followed the time until the spontaneously emitted phonon first reflected from the end of the system. We pointed out that eventually the first emitted phonons would reflect from the end and interfere with phonons emitted later. Thus simulation of the stationary SG quasimode should have a sufficiently long system that there are no reflections during the time of observation or equivalently for a fixed length L the time of observation should be less than (L/c) where c is the speed of sound.

In Fig. 2 of Ref. [3] the length of the system was $L=100$. They followed the time development of the width $l(t)$ for 2500 s. The round trip time of a phonon emitted by a stationary SG soliton at one end reflect and go back to the stationary SG soliton is 200 s. Thus during their 2500 s observation time there were phonons that made more than twelve trips that would interfere with phonons emitted earlier and later. The quasimode lifetime is 500 s. So phonons could be emitted, travel to the end of the system, reflect and be reabsorbed by the still excited quasimode. Consequently, during their 2500 s simulation time, phonons are continuously being emitted, interfering with previously emitted phonons reflecting from the ends of the system and sometimes being absorbed by the stationary SG soliton at the end of the system. Consequently, the simulation of Fig. 2 in Ref. [3] should show a very complicated interference pattern with multiple time scales but with a period of 200 s playing a prominent role, which is precisely what they observe. If they had taken a much longer system or had just simulated for times up to 200 s, they would have verified the existence of the SG quasimode. Actually, the first 200 s of their simulation of the width $l(t)$ gives a very good representation of the SG quasimode. Thus the flawed design of their simulation in Fig. 2 is the cause of their complicated interference pattern.

Once again, in their second numerical solution the authors of Ref. [3] ignore the phonon dressing which gives rise to the SG quasimode. In Ref. [1] we calculated and simulated the phonon dressing which shows how the dressing decays as the quasimode emits phonons during its lifetime while the slope decays from $\Gamma(0)$ to the constant slope Γ_0 and $\dot{\Gamma}(0)$ decays to zero. Also, as in their first numerical simulation, the second numerical solution is for an appreciably discrete phonon system while our derivations and simulations were for the continuum SG equation. Here it is interesting to observe that the qualitative behavior of the appreciably discrete SG system is similar to our analytic calculations and simulations for the continuum SG.

V. DISCUSSION

The authors of Ref. [3] performed two numerical investigations for which they state "we show that neither intrinsic internal modes nor quasimodes exist in contrast to previous reports" referring in particular to our Ref. [1]. In Sec. III we proved that the SG quasimodes that we had derived analytically and verified by simulation could not possibly be observed by their first numerical investigation. The reason is that in order to observe the SG quasimode it must first be created and then observed during its finite lifetime. We proved in Sec. III that an ac driver cannot create a SG quasimode and thus their ac absorption numerical experiment could not possibly observe the SG quasimode but could only measure the phonon absorption spectra of their discrete SG phonon eigenmodes. Our derivations were for a force free, stationary, continuum SG soliton. However, their first numerical investigation is for an ac driven discrete SG soliton. They measured numerically the discrete SG spectrum. However, the phonon spectrum of the SG plays absolutely no role in our analysis, it does not appear in any of our derivations and is irrelevant to our results. The SG quasimode comes from the solution of the coupled continuum equations for the slope of the kink, $\Gamma(t)$, and the phonon dressing for $\chi(t)$. Consequently, their first numerical investigation has no relevance to the existence or nonexistence of the SG quasimode.

In their second numerical investigation of the SG equation they started a discrete stationary SG with an initial rate of change of the slope $\dot{\Gamma}(0)\neq 0$ and $\Gamma(0)\neq 0$. In Ref. [1] we considered three such cases except our derivations and simulations were for the continuum SG and for initial values $\delta\dot{\Gamma}(0)$ of 0.01 and 0.001 and $\delta\Gamma_0=0.1$. The quasimode we derived analytically and verified by simulation was for a linear mode. However, they actually observed the quasimode in the first 200 s of their simulation. They however took a system too short for the length of time they followed the simulations. Consequently, they obtained a very complicated phonon interference pattern due to the multiple phonon interferences due to the earlier emitted phonons interfering with phonons emitted earlier and later because of the multiple reflections of the phonons from the ends of the system. In addition, there were multiple absorptions and reemissions of the phonons with the stationary soliton. If they had increased their system from $L=100$ to $L=400$ and followed the simulation for $t=500$ s instead of their $t=2500$ s they would have obtained essentially the same diagram we obtained for $\Gamma(t)$.

The authors of Ref. [3] state their theoretical analysis of our paper is based on their two cc equations

$$\frac{dP}{dt} = -q\epsilon\sin(\delta t+\delta_0), \tag{17}$$

where

$$P(t) \equiv M_0 l_0 \dot{X} l^{-1}(t),$$

and

$$\alpha(\dot{l}^2 - 2l\ddot{l}) = \frac{l^2}{l_0^2}\left(1+\frac{P^2}{M_0^2}\right) - 1. \tag{18}$$

Their width variable $l(t)/l_0$ is essentially the inverse of our variable $\Gamma(t)$. Since our $X(t)\equiv 0$ their variable $X(t)$ should be identically zero and have no relevance to any of our derivations and verifying simulations. They obtained the width $l(t)$ in their numerical solution of the discrete SG in their Fig. 2, which we discussed in detail in Sec. III. Furthermore, their Eq. (3) for $l(t)$ is incorrect because it contains none of the many terms proportional to $\chi(t)$ that appear in the exact equation of motion for $\Gamma(t)$ in Eq. (8) which are necessary for the existence of the quasimode. Consequently, their two equations of motion for $P(t)$ and $l(t)$ which they state are the "basis of their theoretical analysis" of Ref. [3] have no relevance to our analytic derivation and confirming simulations.

One surprising aspect of Ref. [3] is the lack of any mention or discussion of the continuum states χ of the SG equation in the presence of the SG soliton that are responsible for the existence of the SG quasimode. The solution for χ derived in Eq. (11) of Ref. [1] constitutes a dynamical dressing of the sine-Gordon soliton due to the oscillation of $\Gamma(t)$.

Several times in Ref. [3] the authors compare the SG quasimode with the ϕ^4 equation internal mode, which is an exact eigenmode of the linearized ϕ^4 equation whose eigenfrequency is in the phonon gap. They state that since they can observe the ϕ^4 eigenmode in absorption but cannot observe the SG quasimode in absorption, this proves the SG quasimode does not exist. However, a quasimode is not an eigenmode. A quasimode can be observed in emission but cannot be observed in absorption unless it is first created and then observed within its finite lifetime. In Ref. [1] we proved that the SG quasimode cannot be excited by the ac field used in Ref. [3].

Finally, in spite of the fact that all our derivations and verifying simulations were done for the continuum SG, the simulations and analysis by the authors of Ref. [3] were done for an appreciably discrete SG equation. In particular they report finding a discrete mode in the phonon gap that was found by Kevrekidis and Jones, Ref. [4]. It is important to observe that none of the discrete simulations are relevant to the exact analytic calculations and simulations of our continuum SG equation.

VI. CONCLUSION

In Ref. [3] the authors did not make a simple reference to or comment upon the exact analytic solutions and their verifying simulations of the SG quasimode in Ref. [1]. Furthermore, they never mention the continuum SG states χ that interact with the SG soliton to form the SG quasimode. The authors of Ref. [3] performed two numerical investigations of the discrete SG equation to attempt to prove the SG quasimode does not exist. The first simulation was an absorption measurement of an ac field, which we proved could not possibly observe the SG quasimode in principle. The second simulation was for an initially deformed discrete SG soliton, which they followed in time. In the first 200 s they actually observed the discrete SG soliton. However, they took a system too short for their 2500 s observation time. Consequently, they observed a complicated phonon interference pattern caused by the multiple interferences between phonons emitted at different times which they incorrectly interpreted as the nonexistence of the SG quasimode instead of the fact they observed the system over twelve times too long for the length of their system.

Recently Kalbermann [5] found new important analytic nonperturbative solutions to the SG equation using the inverse scattering transform method. He states "his solutions agree very closely with the results of Boesch and Willis (our Ref. [1]) in the quasimode regime" as shown in his Fig. 4 of Ref. [5]. Also, Kalbermann in Ref. [5] points out "a probable source of error in the numerical calculations of Quintero *et al.*" [3]. The numerical calculation he refers to in his Ref. [5] is the same numerical investigation which we labeled the second numerical investigation in Sec. IV of the present paper. His explanation is essentially identical to the explanation we give in Sec. IV of the present paper.

In their conclusion Quintero *et al.* [3] provided an explanation of our results. They state "the resonance observed by Boesch and Willis took place in fact with the lowest frequency phonon in the presence of a moving kink and not with any internal quasimode." Their argument is, since the kink is moving, there is a Doppler shift, i.e.

$$\bar{\omega}_k = \frac{\omega_k - ku(0)}{(1-u^2(0))^{1/2}},$$

where

$$\omega_k = (1+k^2)^{1/2}.$$

However, all our analytic calculations and verifying simulations are for a stationary kink, i.e., $\dot{X}(t) \equiv 0$ so the kink is not moving and thus the Doppler shift is identically zero. Therefore their explanation that our results are due to a Doppler shift cannot possibly be correct.

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